Chapter 5: Algorithms and Heuristics

CS105: Great Insights in Computer Science
• Let’s talk about how many syllables we sing given a song of a certain type as the number of verses grows.

• In general, we’re interested in the number of syllables as a function of $n$, the number of verses.
Generalized Dreidel Song

1. I had a little dreidel
I made it out of clay
And when it's dry and ready
Oh dreidel I shall play.

Chorus:
Oh dreidel dreidel dreidel
I made it out of clay
And when it's dry and ready
Oh dreidel I shall play.

2. I had a little dreidel
I made it out of plastic
If someone steals my dreidel
I'll do something very drastic.

Chorus

3. I had a little dreidel,
I made it out of glass
My mom said when I spin it,
to spin it on the grass.

Chorus

4. I had a little dreidel,
I made it out of chocolate,
but when I went to spin it,
it had melted in my pocket.

Chorus

5. I had a little dreidel,
I made it out of wood,
and when I went to spin it,
it spun just like it should.

Chorus

6. I had a little dreidel,
I made it out of ice,
but when I went to spin it,
it melted...that's not nice!!

Chorus

7. I had a little dreidel,
I made it out of mud,
and when I went to spin it,
it fell down with a thud.

Chorus

8. I had a little dreidel,
I made it out of tin,
I made it kind of crooked,
and so I always win.

Chorus
## Counting Syllables

<table>
<thead>
<tr>
<th>verses</th>
<th>syllables</th>
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<th>syllables</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>51</td>
<td>5</td>
<td>271</td>
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<tr>
<td>2</td>
<td>109</td>
<td>6</td>
<td>323</td>
</tr>
<tr>
<td>3</td>
<td>161</td>
<td>7</td>
<td>375</td>
</tr>
<tr>
<td>4</td>
<td>219</td>
<td>8</td>
<td>432</td>
</tr>
</tbody>
</table>
• Total syllables roughly, $T(n) = 54 \times n$. 
Old Macdonald had a farm, E-I-E-I-O
And on his farm he had a chick, E-I-E-I-O
With a "cluck, cluck" here and a "cluck, cluck" there
Here a "cluck" there a "cluck"
Everywhere a "cluck-cluck"
With a "neigh, neigh" here and a "neigh, neigh" there
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Old Macdonald: Verse 4

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Old Macdonald had a farm, E-I-E-I-O

25 syllables
Old Macdonald had a farm, E-I-E-I-O
And on his farm he had a chick, E-I-E-I-O

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25 syllables
12 syllables
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25 syllables
22 syllables
12 syllables
Old Macdonald: Verse 4

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Old Macdonald had a farm, E-I-E-I-O

25 syllables
22 syllables
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22 syllables
12 syllables
37 + 4 x 22
= 125 syllables
Old Macdonald: Verse 4

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</table>

Total syllables:

\[37 + 4 \times 22 = 125\] syllables

4-verse song:

\[(37 + 1 \times 22) + (37 + 2 \times 22) + (37 + 3 \times 22) + (37 + 4 \times 22) = 368\] syllables
## Counting Syllables

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<td>5</td>
<td>515</td>
</tr>
<tr>
<td>2</td>
<td>140</td>
<td>6</td>
<td>684</td>
</tr>
<tr>
<td>3</td>
<td>243</td>
<td>7</td>
<td>875</td>
</tr>
<tr>
<td>4</td>
<td>368</td>
<td>8</td>
<td>1088</td>
</tr>
</tbody>
</table>
Plotting Syllables

verses: $n$

syllables
Plotting More Syllables

Total syllables?
• Verse $i$ has $37 + 22 \times i$ syllables.

• Song with $n$ verses:

$$(37 + 1 \times 22) + (37 + 2 \times 22) + \ldots + (37 + n \times 22)$$
• Verse \( i \) has \( 37 + 22 \times i \) syllables.

• Song with \( n \) verses:

\[
(37 + 1 \times 22) + (37 + 2 \times 22) + \cdots + (37 + n \times 22)
\]
Summing Syllables

- Verse $i$ has $37 + 22 \times i$ syllables.
- Song with $n$ verses:

$$37n + (1 + 2 + \ldots + n) \times 22$$
Sum of $n$ Integers

- Old McDonald with $n$ verses:
  
  $37n + 22 \times (1 + 2 + \ldots + n) =$
Sum of $n$ Integers

- Old McDonald with $n$ verses:
  
  $$37n + 22 \times (1 + 2 + \ldots + n) =$$
Sum of $n$ Integers

$$1 + 2 + ... + 6 =$$

- Old McDonald with $n$ verses:
  $$37n + 22 \times (1 + 2 + ... + n) =$$
**Sum of $n$ Integers**

1 + 2 + ... + 6 =

$$(6 \times 7) / 2 = 21$$

- Old McDonald with $n$ verses:

$$37n + 22 \times (1 + 2 + ... + n) =$$
Sum of $n$ Integers

$\text{1 + 2 + ... + 6} = (6 \times 7) / 2 = 21$

$\text{1 + 2 + ... + n} =$

• Old McDonald with $n$ verses:

\[37n + 22 \times (1 + 2 + ... + n) =\]
Sum of $n$ Integers

- Old McDonald with $n$ verses:
  \[37n + 22 \times (1 + 2 + \ldots + n) =\]

\[
1 + 2 + \ldots + 6 = (6 \times 7) / 2 = 21
\]

\[
1 + 2 + \ldots + n = n \times (n+1) / 2
\]
Sum of $n$ Integers

$1 + 2 + \ldots + 6 = \frac{(6 \times 7)}{2} = 21$

$1 + 2 + \ldots + n = \frac{n \times (n+1)}{2}$

• Old McDonald with $n$ verses:

$37n + 22 \times (1 + 2 + \ldots + n) = 11 \ n^2 + 48 \ n$
N Bottles of Beer

- Verse 99:
  99 bottles of beer on the wall.
  99 bottles of beer.
  If one of those bottles should happen to fall.
  98 bottles of beer on the wall.

- Verse i:
  29 syllables + 2 x syllables in i + syllables in i-1.

- Syllables in i?
  - Roughly the number of digits in i.
  - Very slow growing function... by googol, only reaches 101.
Logarithms

- $\lg 100 = 2$
- $\lg 10000 = 4$
- $\lg x$: roughly the number of times you can divide $x$ by 10 before you reach 1 or less.

Other logs:
- $\ln$ is the natural logarithm (base $e$)
- $\log$ is base 2 logarithm: number of times you can halve before reaching 1 or less.
Whole Song

- So, syllables in verse $i$ of $n$ Bottles of Beer:
  - Approximately, $29 + 3 \lg i$.

- $n$ verses: $29n + 3 (\lg 1 + \lg 2 + ... + \lg n)$
  - $\lg 1 + \lg 2 + ... + \lg n$ is approximately $n \lg n$
  - Approximately, $3(n \lg n) + 29n$. 
• So, syllables in verse $i$ of $n$ Bottles of Beer:
  - Approximately, $29 + 3 \log i$.

• $n$ verses: $29n + 3(\log 1 + \log 2 + \ldots + \log n)$
  - $\log 1 + \log 2 + \ldots + \log n$ is approximately $n \log n$
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So, syllables in verse \( i \) of \( n \) Bottles of Beer:
- Approximately, \( 29 + 3 \lg i \).

\( n \) verses:
- \( 29n + 3 \left( \lg 1 + \lg 2 + \ldots + \lg n \right) \)
- \( \lg 1 + \lg 2 + \ldots + \lg n \) is approximately \( n \lg n \)
- Approximately, \( 3 \left( n \lg n \right) + 29n \).
On the tenth day of Christmas, my true love sent to me
Ten lords a-leaping,
Nine ladies dancing,
Eight maids a-milking,
Seven swans a-swimming,
Six geese a-laying,
Five golden rings,
Four calling birds,
Three French hens,
Two turtle doves,
And a partridge in a pear tree.

Verse 10:
Verse $i$:

11 syllables + (4 + syllables in $i$) + (4 + syllables in $i-1$) + (4 + syllables in $i-2$) + ... + (4 + syllables in 1).

Approx., $11 + 4 \cdot i + \lg 1 + \lg 2 + ... + \lg i$

Approx., $11 + 4 \cdot i + i \cdot \lg i$. 
Whole Song

- $n$ verses:
  \[(11 + 4 + \log 1) + (11 + 8 + 2 \log 2) +
  (11 + 12 + 3 \log 3) + ... + (11 + 4n + n \log n)\]
- Approx., $11n + 2n(n+1) + .46n(n+1) \log n$.
- Approx., $.46n^2 \log n + 2n^2 + .46n \log n + 13n$. 
Different Growth Rates

- With these constants, \( n^2 \lg n \) has fastest growth, then quadratic, then \( n \lg n \), then linear.

- For big \( n \), always the same order regardless of the constants!

- Leads to the notion of “Big O”.

0.572 \( n^2 \lg n \)

0.4 \( n^2 \) (Quadratic)

2.86 \( n \lg n \)

2 \( n \) (Linear)
**Big O**

- Formally, big O is a notation that denotes a class of functions all of which are upper bounded asymptotically.

- In practice, however, it gives us a way of ignoring constants and low-order terms to cluster together functions that behave similarly.

\[ 17n + 91.2 \log(n) + n^{1/2} \]
Big O

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\[ O(n) \]
Common Growth Classes

- Linear: $O(n)$
  - Dreidel
  - Clementine
- Quadratic: $O(n^2)$
  - An Old Lady
  - Old Macdonald
  - There Was a Tree
- $O(n \log n)$
  - $N$ Bottles of Beer
  - $N$ Little Monkeys
- $O(n^2 \log n)$
  - $N$ Days of Christmas
  - Who Knows $N$?
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*constant size verse*

*each verse contains the next higher number*
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  - Who Knows $N$?

constant size verse

each verse contains the next higher number

each verse lists one more number than the previous

each verse a constant size larger than the previous
Another Visualization

linear (O(n))

O(n log n)

quadratic (O(n^2))
Non-Classical Songs

• As far as I know, classical songs are all linear ($O(n)$), quadratic ($O(n^2)$), $O(n \log n)$, and $O(n^2 \log n)$.

• Nevertheless, I can make up a few more songs to demonstrate a few other important growth rates.
My kids used to play this game: “I can count up to 100. One, two, skip-a-few, 99, 100!”. Or “One, two, skip-a-few, 999, 1000!”.

Number of syllables to “skip count” to $n$?

- $5 + 2 \lg n$: This song is $O(\lg n)$.

With exponential notation: “One, two, skip-a-few, $10^{100}-1$, $10^{100}$. Now, the syllables depend on the number of digits: $O(\lg \lg n)$. 
• On the flip side, consider a song in which verse $i$ consists of singing all the numbers with exactly $i$ digits.

• Now, a song with $n$ verses is $O(10^n)$.

• This is an exponential growth. Something I’d like to say a bit more about.
Exponential Growth

- ipods.
- Computer speed: Moore’s Law.
- World Population.
- Bacterial growth (while the food lasts).
- Spam.
Pet Peeve Alert

• Because exponential growth rates are so common, the phrase has entered the public lexicon.

• Not always properly... Many people seem to use it to mean “a lot more”, which doesn’t really make sense.

• Let’s learn to recognize the proper use, ok?
Which Are Correct?

- **Source: Newsweek.**

- The country desperately needs to upgrade its roads and seaports, and to **exponentially** increase agricultural and manufactured exports.
  
  - \( \text{exports}(t) = 10^t \)

- **Exponentially** less expensive than a 20-hour flight to the Bushveld of South Africa or the remote rain forests of Costa Rica, domestic safaris can be nearly as exciting—and far more accessible for families with kids.

- The demands on an organization to carry out [multiple] attacks like that probably increase **exponentially**. In other words, to carry out four simultaneous bombings is more difficult than simply just four times the difficulty of carrying out one bombing.
  
  - \( \text{difficulty}(\text{targets}) = 10^{\text{targets}} \)

- But a small number of others, knowing that their chance of success with PGD is **exponentially** better, are becoming pioneers in the newest form of family planning.
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- difficulty(targets) = 10^{targets}

But a small number of others, knowing that their chance of success with PGD is **exponentially** better, are becoming pioneers in the newest form of family planning.
• Demand for IVF treatments, which climbed **exponentially** during the past 20 years, has plateaued.
  
  \[
  \text{demand}(t) = 10^t
  \]

• Consequently, an unintended but **exponentially** growing number of middle-class Americans is being affected.
  
  \[
  \text{affectedpeople}(t) = 10^t
  \]

• I have been on television for almost 12 years, and in that relatively short time I've seen the medium change **exponentially**.

• Now in the tsunami's aftermath, global health experts worry that the dangerous microbes already lurking in underdeveloped regions of Asia will spread **exponentially**, pushing the tsunami's enormous death toll even higher.
  
  \[
  \text{affectedArea}(t) = 10^t
  \]

• Injury rates [for cheerleaders] are "**exponentially** higher for a flier than for a footballer," says NCCSI's Robert Cantu.
Algorithm Analysis

- Now, that we have a sense of how various quantities grow as a function of other quantities.

- Let’s apply this idea to analyzing our sock sorters.

- For each algorithm, how does the number of reaches into the laundry basket grow as a function of the number of pairs of socks $n$?
Random Probability Facts

• Since the sock sorting setting involves probabilities, it helps to review a few facts.

• If an event happens on each try with probability $p$, we’d expect $1/p$ tries (on average) before we’re successful. Example: Average number of die rolls before getting a 3 (probability 1/6) is 6.

• If we look through a randomly ordered list of length $n$ for an item on the list, on average we’ll need to look through $1/2 (n/2)$ of the list.
Analyzing Sock Sorting

- How many socks does sockA take out of the basket to sort 50 pairs of socks?

- sockA: choose a random pair. Return to basket if no match.

- # of socks removed before a pair is found?

- Probability of a match is 1/99.

- Number of tries before match found? 99, on average.

- Each of the 99 tries removes two socks, so 198 for the first pair, on average.
So, how many socks removed to find the first pair given $n$ pairs in the basket? 2($2n-1$) = $4n-2$.

Now, there are $n-1$ pairs left. Finding the second pair will take $4(n-1)-2 = 4n-6$ sock removals.

When there is one pair left, it takes 2 sock removals.

Total
= $2 + 6 + 10 + ... + 4n-2$
= $4(1+2+...+n)-2n$
= $4 n(n+1)/2 - 2n$
= $2n^2$.

So, $O(n^2)$ algorithm.
Intuitive Analysis

- Since the time to find each pair is proportional to the number of pairs left, the total amount of time until all pairs are found is roughly $n^2$.

- sockC is the same, except the time is halved. Still order $n^2$. 

Graph: 
- x-axis: pairs completed 
- y-axis: # of socks removed 
- Quadratic behavior ($O(n^2)$)
How about sockB?

- **sockB**: Keep a pile on the table. Grab a sock and check if its mate is already out. If not, add it to the pile.

- Since all socks are matched up and no socks are returned to the basket, each sock is removed from the basket precisely once, $2n$ if $n$ pairs.

- So, a O($n$) algorithm! Linear, order $n$, etc.

- No wonder it’s fast.
Algorithm Design Goal

• Not just trying to solve a problem, but solve it well with respect to some goal.

• Best way to the airport?
  - Time?
  - Money?
  - Gas?
  - What else?