

Lower Bounds for Clear Transmissions in Radio Networks

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Abstract. We show new lower bounds for collision-free transmissions in Radio Networks. Our main result is a tight lower bound of $\Omega(\log n \log(1/\epsilon))$ on the time required by a uniform randomized protocol to achieve a clear transmission with success probability $1 - \epsilon$ in a one-hop setting. This result is extended to non-uniform protocols as well. A new lower bound is proved for the important multi-hop setting of nodes distributed as a connected Random Geometric Graph. Our main result is tight for a variety of problems.

Keywords

Ad-hoc network, Broadcast, Contention resolution, Dominating set, Leader election, Lower bound, Maximal independent set, Radio network, Random geometric graphs, Station selection, Sensor network, Wake up, Weak Sensor Model. ACM-class: F.2.2.

1 Introduction

Any network where transmissions may collide needs a protocol for *collision-free transmissions*. Different networks provide different information about collisions. For example, on some hardware, transmitters can distinguish amongst three states at each time step: no transmission, single transmission, and collision, whereas on other hardware, transmitters can not distinguish between no transmission and collisions. In some networks, transmitters know an upper bound on their number. Sometimes, transmitters may not *snoop*, i.e., listen to the channel when not transmitting; whereas at the other extreme, transmitters may only snoop, i.e., they get no information on the channel when they are transmitting. In some networks collisions are transitive. The properties of a shared channel have a profound impact on the protocols usable on such a channel.

Sensor networks are a heavily studied example of a shared-channel network. A sensor network consists of small devices with processing, sensing and communication capabilities. These *sensor nodes* are randomly deployed over an area in order to achieve sensing tasks after self-organizing as a wireless radio network. Sensor nodes

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have strong limitations and operate under harsh conditions. Some of the important limitations of sensor nodes include: lack of collision detection hardware, non-simultaneous transmission and reception, and one channel of communication. We call any such network a Radio Network. Additionally, nodes in sensor networks wake up at arbitrary times. Sensor networks are even more restricted in various ways that will not concern us here. The Radio Network restrictions, along with these further restrictions, are part of the Weak Sensor Model presented in [1].

Algorithms for achieving a *clear*, that is, uncolliding, transmission have been studied in several shared-channel contention settings¹. Hayashi, Nakano and Olariu [3] presented the first $O(\log^2 n)$ algorithm for clear transmission with high probability in one-hop Radio Networks. A $O(\log n \log(1/\epsilon))$ -time algorithm for a clear transmission with probability $1 - \epsilon$ in a one-hop Radio Network was introduced in [2]. Strikingly, when $\epsilon = 1/n$ the same time bound can be obtained for the much more complicated problem of computing a Maximal Independent Set (MIS) in the multi-hop Weak Sensor Model [7]. Kushilevitz and Mansour [5] proved the first lower bound of $\Omega(\log n)$ on the expectation of the running time of any randomized algorithm for clear transmissions in radio networks. A lower bound of $\Omega(\log n \log(1/\epsilon)/(\log \log n + \log \log(1/\epsilon)))$ for achieving a clear transmission with probability $1 - \epsilon$ in a one-hop, globally-synchronized Radio Network was proved in [4]. The latter lower bound is tighter than the previous one if ϵ is $o(1/\log n)$.

Our results: The gap between the lower bound for achieving something so simple as a clear transmission and upper bounds for more complicated problems such as MIS is tantalizingly narrow: respectively $\Omega(\log^2 n / \log \log n)$ and $O(\log^2 n)$, when ϵ is $\Theta(1/n^c)$. In this paper, we close this gap by proving a stronger lower bound: it takes time $\Omega(\log n \log(1/\epsilon))$ to solve the problem of achieving a clear transmission with probability $1 - \epsilon$ in a one-hop setting, which implies, for example, the $\Omega(\log n)$ lower bound on the expectation of any randomized algorithm for clear transmission. Our lower bounds apply to any network with the following characteristics:

- *Shared channel of communication:* All nodes communicate with their neighbors using broadcasts that are transmitted on a shared channel.
- *Lack of a collision detection mechanism:* Nodes do not have the ability to distinguish between a collision on the channel or lack of a transmission.
- *Non-simultaneous transmission and reception:* Nodes cannot snoop on the channel while transmitting.
- *Local synchronization:* Time is assumed to be divided into slots and all nodes have the same clock frequency.
- *Adversarial wake-up schedule:* Nodes are woken up by an adversary.

Indeed, we will prove our lower bound with the following weak adversary: the adversary may choose an $i \in [1, \log n]$, and 2^i nodes wake up at time 0. Our techniques

¹ In a one-hop Radio Network, the clear transmission problem is equivalent to the so-called *wake-up* and *leader election* problems. These problems differ in multi-hop networks, although, a clear transmission is necessary to achieve wake-up and leader election since, indeed, a clear transmission is necessary to solve any problem on a Radio Network. Since we are interested in lower bounds, we will cite bounds for the clear transmission problem in previous papers, even when the bounds were originally stated for the other problems.

also give us a lower bound of $\Omega(\log \log n \log(1/\epsilon))$ on clear transmissions in the well-studied case of sensor nodes distributed uniformly at random with enough nodes to ensure connectivity, and thus for more complicated problems such as MIS. There was no non-trivial lower bound known for this problem, and the best upper bound known is $O(\log^2 n)$ with high probability, proved for the more complicated problem of sensor network initialization in the Weak Sensor Model [1].

1.1 Roadmap

In Section 2, we show the main lower bound, for uniform protocols in one-hop networks. In Section 3, we extend this result in two ways. We show a lower bound for nonuniform protocols, and a lower bound for nodes distributed geometrically.

2 Uniform Protocols in One-Hop Radio Networks

In this section, we prove a lower bound on randomized uniform protocols and extend the result to nonuniform protocols in Section 3.

We first define what the clear transmission problem is in the one-hop setting. The nodes are all connected to a common broadcast channel and each transmission is available for snooping to all non-transmitting nodes. The connectivity of the nodes can be modelled as a clique. In this case we assume that all nodes know an upper bound on the number of their neighbors. In this setting, a clear transmission is achieved if exactly one node transmits in a time slot.

As explained in Section 1, we prove our lower bounds under the assumption of the existence of a weak adversary that, at a given time, wakes up (i.e. turns on) some subset of nodes. We call them *active* nodes. Upon waking up, the active nodes start the execution of a protocol to achieve a clear transmission. All non-active nodes do not participate in the protocol.

We define a *randomized uniform protocol* for clear transmission to be a sequence p_1, p_2, \dots where each node transmits with probability p_ℓ in the ℓ^{th} time step after waking up. Given our adversary, this means that all active nodes transmits with same probability as each other in each time slot.

We seek a lower bound on the number of time-slots required to achieve a clear transmission with probability $(1 - \epsilon)$. We simplify the analysis in two ways. First, we further weaken the adversary by requiring that the number of nodes participating can only be one of $\{2^i | 0 \leq i \leq \log_2 n\}$. Secondly, we assume that all $p_\ell \in \{2^{-j} | 1 \leq j \leq \log_2 n\}$. If this assumption is not true of a particular algorithm A , we can always produce an algorithm A' from A by replacing one attempt in A by a constant number of attempts in A' where the probabilities of transmission in A' have been rounded off to the closest power of $1/2$.

One of the principal benefits of our weak adversary is that, the probability P_ℓ of a clear transmission by time ℓ is the same for any permutation of p_1, p_2, \dots, p_ℓ . Therefore, we need not bother with what order the steps are taken in, but only how many times the protocol fires with each probability.

Let t_j be the number of time-slots that nodes are transmitting with probability 2^{-j} . Let p_{ij} denote the probability that 2^i nodes fail to clear when they all transmit with probability 2^{-j} . Thus we know that:

$$\begin{aligned} p_{ij} &= 1 - 2^i \frac{1}{2^j} \left(1 - \frac{1}{2^j}\right)^{2^i - 1} \\ &= 1 - 2^{i-j} (1 - 2^{-j})^{2^i - 1} \end{aligned}$$

The total probability of failure for any number of active nodes, 2^i , needs to be bounded by:

$$\begin{aligned} \prod_j p_{ij}^{t_j} &\leq \epsilon \\ \iff \sum_j t_j \ln(p_{ij}) &\leq \ln(\epsilon). \end{aligned}$$

A lower bound is achieved by minimizing the total number of time-slots needed to satisfy the previous constraints. This can be formulated as the following *primal* linear program:

$$\begin{aligned} &\text{Minimize } \mathbf{1}^T \mathbf{t}, \\ &\text{subject to:} \\ &\quad \mathbf{P} \mathbf{t} \geq \epsilon \\ &\quad \mathbf{t} \geq \mathbf{0} \\ &\text{where:} \\ &\quad \mathbf{t} \triangleq [t_j], \\ &\quad \epsilon \triangleq [-\ln(\epsilon)], \\ &\quad \mathbf{P} \triangleq [-\ln(p_{ij})], \end{aligned}$$

which yields the following *dual*:

$$\begin{aligned} &\text{Maximize } \epsilon^T \mathbf{u}, \\ &\text{subject to:} \\ &\quad \mathbf{P}^T \mathbf{u} \leq \mathbf{1} \\ &\quad \mathbf{u} \geq \mathbf{0}. \end{aligned}$$

The primal linear program has a finite minimum solution, and hence its dual has a finite maximum solution. The value of the objective function for every feasible solution of the dual is a lower bound on the minimum value of the objective function for the primal. Thus any feasible solution for the dual will give a lower bound on the number of time-slots required to achieve a clear transmission with failure probability ϵ .

Suppose that the j^{th} row, \mathbf{P}_j^T , of \mathbf{P}^T has the maximum row sum, and let $r(\mathbf{P}^T) = \mathbf{P}_j^T \mathbf{1}$. Now we set $\mathbf{u} = [1/r(\mathbf{P}^T)]$. This value of \mathbf{u} satisfies all constraints of the dual.

The value of the objective function of the dual is simply $\epsilon^T \mathbf{u}$. To obtain the value of the objective function of the dual we need to find the row of \mathbf{P}^T with the largest row sum which is the same as the column of \mathbf{P} with the largest column sum.

Lemma 1. *The trace of every column vector of the constraint matrix \mathbf{P} of the primal is in $O(1)$.*

Proof. We begin by stating the following useful inequality [6, §2.68]:

$$e^{-x/(1-x)} \leq 1 - x \leq e^{-x}, 0 < x < 1. \quad (1)$$

The sum of the elements of a column j of \mathbf{P} is:

$$\begin{aligned} S_j &\leq \sum_i -\ln(1 - 2^{i-j}(1 - 2^{-j})^{2^i-1}) \\ &\leq \sum_i -\ln\left(\frac{e^{-2^{i-j}(1-2^{-j})^{2^i-1}}}{(1-2^{i-j}(1-2^{-j})^{2^i-1})}\right) \quad (\text{By Inequality 1}) \\ &= \sum_i \frac{2^{i-j}(1 - 2^{-j})^{2^i-1}}{1 - 2^{i-j}(1 - 2^{-j})^{2^i-1}}. \end{aligned}$$

Let $y_{ij} \triangleq 2^{i-j}(1 - 2^{-j})^{2^i-1}$.

$$\begin{aligned} S_j &= \sum_i \frac{y_{ij}}{1 - y_{ij}} \\ &\leq \sum_i \frac{y_{ij}}{1 - y_{max}} \quad (\text{where } y_{max} = \max_{ij} \{y_{ij}\}). \end{aligned}$$

Now we derive an upper bound on y_{max} :

$$\begin{aligned} y_{max} &= \max_{ij} y_{ij} \\ &= \max_{ij} 2^{i-j}(1 - 2^{-j})^{2^i-1} \\ &\leq \max_{ij} 2^{i-j} e^{-2^{i-j} + 2^{-j}} \quad (\text{By Inequality 1}) \\ &\leq \max_{ij} \sqrt{e} \frac{2^{i-j}}{e^{2^{i-j}}} \quad (\because j \geq 1) \\ &\leq \frac{1}{\sqrt{e}} \quad (\text{The function is maximized, when } i = j). \end{aligned}$$

Therefore:

$$S_j \leq \frac{\sqrt{e}}{\sqrt{e} - 1} \sum_i y_{ij}$$

We derive an upper bound on the right hand side sum.

$$\begin{aligned}
\sum_i y_{ij} &= \sum_i 2^{i-j} (1 - 2^{-j})^{2^i - 1} \\
&\leq \sum_i 2^{i-j} (e^{-2^{-j}})^{2^i - 1} \text{ (By Inequality 1)} \\
&= \sum_i 2^{i-j} e^{-2^{i-j} + 2^{-j}} \\
&\leq \sqrt{e} \left(\sum_{i \geq j} 2^{i-j} e^{-2^{i-j}} + \sum_{i < j} 2^{i-j} e^{-2^{i-j}} \right) (\because j \geq 1) \\
&\leq \sqrt{e} \left(\sum_{k \geq 0} 2^k e^{-2^k} + \sum_{k \geq 1} 2^{-k} e^{-2^{-k}} \right) \\
&\leq \sqrt{e} \left(\sum_{k \geq 0} 2^k e^{-2^k} + \sum_{k \geq 1} 2^{-k} \right) \\
&\in O(1) \text{ (Because both the sums are bounded by a constant)} \\
\implies S_j &\in O(1).
\end{aligned}$$

Theorem 1. *Every uniform randomized algorithm to achieve a clear transmission with probability $1 - \epsilon$ in a one-hop Radio Network requires $\Omega(\log n \log(1/\epsilon))$ time-slots.*

Proof. From lemma 1, we know that $r(P^T) \in O(1)$, then $\epsilon^T \mathbf{u} = [-\ln(\epsilon)] \cdot [1/\mathbf{P}_{max}^T \mathbf{1}] \in O(\log n \log(1/\epsilon))$. From this we can conclude that the dual linear program has a feasible solution with objective function evaluating to $\Omega(\log n \log(1/\epsilon))$. Since we showed earlier that the solution to the primal linear program gives a lower bound on the number of time-slots required to achieve a clear transmission with probability $1 - \epsilon$, the statement of the theorem holds.

3 Extensions

In this section, we show how to obtain lower bounds for nonuniform protocols and for geometric distributions of nodes.

3.1 Randomized Nonuniform Protocols in One Hop Radio Networks

In this section we prove our lower bound for the case in which processors may run different algorithms using their unique ID's to break symmetry. We call this a *nonuniform protocol*. Recall that we model a randomized protocol to achieve a clear transmission as a schedule, or temporal sequence, of probabilities of transmission such that, at time slot i an active node transmits with probability p_i . In the case of the randomized uniform protocols, we assume that nodes either have no ID or the protocol does not make use of

it to break symmetry. Then, given that no information can be obtained from a shared-channel before a clear transmission, all active nodes transmit with the same probability in the same time slot. On the other hand, if nodes have unique ID's, they may use different schedules of probabilities of transmission and achieve a clear transmission faster. We prove in this section that in fact having unique ID's does not help.

As in [5], we prove our lower bound by showing a reduction from a nonuniform protocol to a uniform one. We first state our result formally.

Theorem 2. *Every randomized nonuniform protocol to achieve a clear transmission with probability $1 - \epsilon$ in a one-hop Radio Network requires $\Omega(\log n \log(1/\epsilon))$ time slots.*

Proof. For the sake of contradiction, assume that there exists a randomized nonuniform protocol \mathcal{A} that achieves a clear transmission with probability $1 - \epsilon$ in T time slots, where $T \in o(\log n \log(1/\epsilon))$. Then, we can define a randomized uniform protocol \mathcal{A}' that achieves the same running time as follows.

For each node

*Choose uniformly at random an integer $i \in [1, n^2/\epsilon]$.²
Simulate protocol \mathcal{A} using i as ID.*

Each node running the protocol \mathcal{A}' obtains a unique ID with probability at least $1 - \epsilon$. This is true because the probability that some pair of nodes chooses the same ID is ϵ/n^2 and there are $\binom{n}{2}$ possible pairs. Given that the random choice of the ID can be done in constant time, the protocol \mathcal{A}' is a randomized uniform protocol that achieves a clear transmission with probability $1 - 2\epsilon$ in $o(\log n \log 1/\epsilon) = o(\log n \log 1/2\epsilon)$ time slots, which is a contradiction with Theorem 1.

3.2 Randomized Protocols for Geometrically Distributed Nodes

We begin with some preliminaries on geometrically distributed nodes, before getting to the lower bound. In the Random Geometric Graph Model $\mathcal{G}_{n,r,\ell}$, n nodes are distributed uniformly at random in $[0, \ell]^2$, and nodes are connected by an edge iff they are at Euclidean distance at most r , the *connectivity radius*. The node density depends on the relative values of n, r and ℓ . A specific instance of $\mathcal{G}_{n,r,\ell}$ is a *Random Geometric Graph (RGG)*, also referred to as $G(n, r, \ell)$. In a $G(n, r, \ell)$, the asymptotic behavior of route stretch is studied as $\ell \rightarrow \infty$ while maintaining sufficient density to preserve connectivity. In this paper we will assume that parameter conditions to ensure connectivity are always satisfied [8].

Here we consider the problem of achieving a clear transmission under the following conditions:

The nodes are connected by a broadcast channel to some subset of nodes and each transmission made by a node is available to its neighbors only, but it can interfere with

² Under the assumptions of the Weak Sensor Model, nodes have only $O(\log n)$ bits of memory. Therefore, this lower bound applies also to sensor networks when $\epsilon \geq 1/n^\gamma$, for some constant $\gamma > 0$.

all transmissions originating in a two-hop neighborhood. The specific case we will derive a lower bound for is the case of nodes consistent with the Weak Sensor Model distributed randomly in the plane with limited transmission range but adequate density to ensure connectivity. The connectivity of the nodes can be modelled as a *Random Geometric Graph (RGG)*. In this case, we assume that nodes know an upper bound on the number of their neighbors with a probability given by the parameter conditions for connectivity.

In this setting, we say that a clear transmission occurred if exactly one node is transmitting and no other nodes within two hops of it are transmitting. Then, the clear transmission problem in a multi-hop setting is solved after every node either produces or receives a clear transmission.

In a $G(n, r, \ell)$ satisfying the connectivity conditions explained previously, the number of nodes contained in any circle of radius $\Theta(r)$ is $\Theta(\log n)$ with high probability. This can be proved by a simple application of the Chernoff-Hoeffding bounds. Then, we complete our lower bounds with the following corollary, which can be obtained as a simple application of Theorems 1 and 2 to this setting.

Corollary 1. *Every randomized protocol to solve the clear transmission problem with probability $1 - \epsilon$ in a Radio Network with geometrically distributed nodes requires $\Omega(\log \log n \log(1/\epsilon))$ time slots, where $\epsilon \geq 1/n^\gamma$ for some constant $\gamma > 0$.*

Proof. Replacing the appropriate density for any one-hop neighborhood in this setting, i.e. $\Theta(\log n)$ instead of n , in theorem 2 the corollary follows.

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